Error analysis for Heun's method

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In class, it is claimed that the error for Heun's method is O(h^3), but some students are interested in how this is derived. This is a detailed derivation using Taylor series for multivariate functions, as may have been seen in your first-year calculus courses. We then give two examples to demonstrate that the error terms found do reflect the actual applications of Heun's method for two examples.

An initial-value problem (IVP) is a differential equation together with an initial condition

$$\begin{aligned} y^{(1)}(t) &= f(t, y(t)) \\ y(t_0) &= y_0 \end{aligned}$$

Heun's method says that we will approximate $y(t_0 + h)$ as follows:

$$\begin{aligned} s_0 &\leftarrow f(t_0, y_0) \\ s_1 &\leftarrow f(t_0 + h, y_0 + hs_0) \\ y(t_0 + h) &\approx y_0 + h\frac{s_0 + s_1}{2} \end{aligned}$$

The error analysis for Heun's method is slightly more involved than that for Euler's, but uses concepts we have already introduced previously. Taking one addition term to the Taylor series, we have

$$\begin{aligned} y(t_0 + h) &= y(t_0) + y^{(1)}(t_0)h + \frac{1}{2}y^{(2)}(t_0)h^2 + \frac{1}{6}y^{(3)}(\tau_1)h^3 \\ &= y_0 + s_0 h + \frac{1}{2}y^{(2)}(t_0)h^2 + \frac{1}{6}y^{(3)}(\tau_1)h^3 \end{aligned},$$

and now we use the approximation that

$$\begin{aligned} y^{(2)}(t_0) &= \frac{y^{(1)}(t_0 + h) - y^{(1)}(t_0)}{h} - \frac{1}{2}y^{(3)}(\tau_2)h \\ &= \frac{y^{(1)}(t_0 + h) - s_0}{h} - \frac{1}{2}y^{(3)}(\tau_2)h \end{aligned}.$$

If we substitute this into the first equation, we get

$$y(t_0 + h) = y_0 + s_0 h + \frac{1}{2}\left(\frac{y^{(1)}(t_0 + h) - s_0}{h} - \frac{1}{2}y^{(3)}(\tau_2)h\right)h^2 + \frac{1}{6}y^{(3)}(\tau_1)h^3$$

and by expanding this and substituting $y^{(1)}(t_0 + h) = f(t_0 + \mathrm{h}, y(t_0 + h))$, we have:

$$y(t_0 + h) = y_0 + \frac{s_0 + f(t_0 + h, y(t_0 + h))}{2}h + \left[\frac{1}{6}y^{(3)}(\tau_1) - \frac{1}{4}y^{(3)}(\tau_2)\right]h^3.$$

The problem, is, however, that we do not know the value of $y(t_0 + h)$, but we do have an approximation of this value with $y_0 + hs_0$ from Euler's method.

To determine the contribution of this error, recall that

$$f(t, y+\varepsilon) = f(t, y) + \frac{\partial}{\partial y} f(t, \upsilon)\varepsilon$$

where $\upsilon \in [y, y+\varepsilon]$, or $f(t, y) = f(t, y+\varepsilon) - \frac{\partial}{\partial y} f(t, \upsilon)\varepsilon$. In this case,

$$\begin{aligned} y(t_0+h) &= y(t_0) + y^{(1)}(t_0)h + \frac{1}{2} y^{(2)}(\tau_3)h^2 \\ &= y_0 + hs_0 + \frac{1}{2} y^{(2)}(\tau_3)h^2 \end{aligned},$$

and therefore

$$y_0 + hs_0 = y(t_0+h) - \frac{1}{2} y^{(2)}(\tau_3)h^2,$$

Substituting this back in, we get that

$$\begin{aligned} s_1 &= f(t_0+h, y_0+hs_0) \\ &= f\left(t_0+h, y(t_0+h) - \frac{1}{2} y^{(2)}(\tau_3)h^2\right) \\ &= f(t_0+h, y(t_0+h)) - \frac{\partial}{\partial y} f(t_0+h, \upsilon)\left(\frac{1}{2} y^{(2)}(\tau_3)h^2\right) \end{aligned}$$

and therefore

$$\begin{aligned} f(t_0+h, y(t_0+h)) &= f(t_0+h, y_0+hs_0) + \frac{1}{2}\frac{\partial}{\partial y} f(t_0+h, \upsilon) y^{(2)}(\tau_3)h^2 \\ &= s_1 + \frac{1}{2}\frac{\partial}{\partial y} f(t_0+h, \upsilon) y^{(2)}(\tau_3)h^2 \end{aligned}.$$

Substituting this back into our equation,

$$y(t_0+h) = y_0 + h\frac{s_0+s_1}{2} + \left[\frac{1}{6} y^{(3)}(\tau_1) - \frac{1}{4} y^{(3)}(\tau_2) + \frac{1}{4}\frac{\partial}{\partial y} f(t_0+h, \upsilon) y^{(2)}(\tau_3)\right]h^3.$$

Thus, we see that the error is $\mathrm{O}(h^3)$.

Example

To demonstrate that this is the correct error formula, we will use the IVP $y^{(1)}(t) = -y(t) + 1$ with $y(0) = 0.7$, and from the solution we can estimate the coefficient of the error term for Heun's method derived above:

$$\left(\frac{1}{6} - \frac{1}{4}\right) y^{(3)}(0) + \frac{1}{4}\frac{\partial}{\partial y} f(0, 0.7)\, y^{(2)}(0) = 0.05,$$

For this IVP, we note that $y(0.1) = 0.72854877458921212805$ and the approximation using Heun's method is 0.7285. The error of the approximation is 0.00004877, and we see that this is very close to $0.05 \times 0.1^3 = 0.00005$. We also note that $y(0.01) = 0.70298504987524958393$ and the approximation using Heun's method is 0.702985. The error of the approximation is 0.00000004988, and we see that this is even closer to $0.05 \times 0.01^3 = 0.00000005$. The exact solution is $y(t) = 1 - 0.3e^{-t}$.

As a second example, we will use the IVP $y^{(1)}(t) = -ty^2(t) + 1$ with $y(1) = 1.3$, we have

$$\left(\frac{1}{6} - \frac{1}{4}\right) y^{(3)}(1) + \frac{1}{4}\frac{\partial}{\partial y} f(1, 1.3)\, y^{(2)}(1) = -0.26471\overline{6}.$$

For this second IVP, we note that $y(1.1) = 1.2318764249005373919$ and the approximation using Heun's method is 1.232155145. The error of the approximation is –0.0002787, and we see that this is very close to $-0.2647 \times 0.1^3 = 0.00026457$. We also note that $y(1.01) = 1.2931055903341764764$ and the approximation using Heun's method is 1.2931058565695. The error of the approximation is –0.0000002662, and we see that this is even closer to $-0.2647 \times 0.01^3 = -0.0000002647$.

Incidentally, the exact solution to the second IVP is

$$y(t) = \frac{\mathrm{Bi}(t)\left(\mathrm{Ai}(1) - 1.3\mathrm{Ai}^{(1)}(1)\right) - \mathrm{Ai}(t)\left(\mathrm{Bi}(1) - 1.3\mathrm{Bi}^{(1)}(1)\right)}{\mathrm{Bi}^{(1)}(t)\left(\mathrm{Ai}(1) - 1.3\mathrm{Ai}^{(1)}(1)\right) - \mathrm{Ai}^{(1)}(t)\left(\mathrm{Bi}(1) - 1.3\mathrm{Bi}^{(1)}(1)\right)},$$

where Ai and Bi are the [Airy functions](#).

An analysis similar to that for Euler's method would show that if we applied Heun's method in multiple steps, the error would be approximately

$$\left[\frac{1}{4}\overline{\frac{\partial f}{\partial y}}\,\overline{y^{(2)}} - \frac{1}{12}\overline{y^{(3)}}\right]\left(t_f - t_0\right)h^2,$$

and thus halving the step size will reduce, on average, the error by a factor of four.